Introduction to Simon-Smith min-max theory

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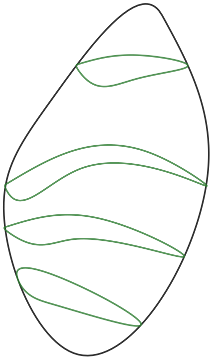
Apart from minimal constructions, min-max arguments have been used in other contexts, like te proof of the Willmore conjecture by **Marques** and **Neves**, constructions of CMC and PMC surfaces, free boundary and capillary problems...

An intuitive idea of Birkhoff's theorem

Given a 2-sphere, we sweep it out by a family of closed curves γ_t, $t \in [0, 1]$.



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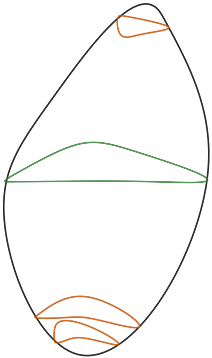

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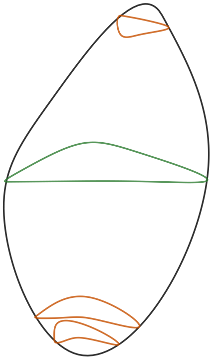

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We expect the length of Γ to satisfy

$$\mathrm{Len}(\Gamma) = \min_{\{\tilde{\gamma}_t\}\in\Lambda} \max_t \mathrm{Len}(\tilde{\gamma}_t),$$

where Λ is the set of all deformations of γ_t.

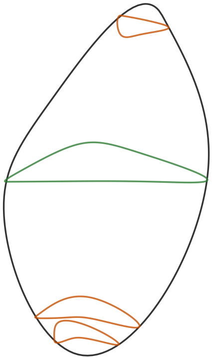

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Table of Contents

Generalized families of surfaces and saturated sets

A family $\{\Sigma_t\}_{t\in[0,1]}$, $\Sigma_t \subset M$, is a **generalized family of surfaces** (GFS) if:

- Σ_t is a surface except for a finite set $t \in \mathcal{T} \subset [0,1]$.
- There exists a finite set $\mathcal{P} \subset M$ such that $\Sigma_t \setminus \mathcal{P}$ is a surface for all $t \in \mathcal{T}$.
- $t \mapsto \mathcal{H}^2(\Sigma_t)$ is continuous.
- $t \mapsto \Sigma_t$ is continuous in the Hausdorff topology.

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A collection of GFS's Λ is a **saturated set** if it is closed under the previous operation.

Inf max and minimizing sequences

Given $\{\Sigma_t\} \in \Lambda$, we define:

$$\mathcal{F}(\{\Sigma_t\}) := \max_{t\in[0,1]} \mathcal{H}^2(\Sigma_t)$$

$$m_0(\Lambda) := \inf_{\Lambda} \mathcal{F} = \inf_{\{\Sigma_t\}\in\Lambda} \max_{t\in[0,1]} \mathcal{H}^2(\Sigma_t)$$

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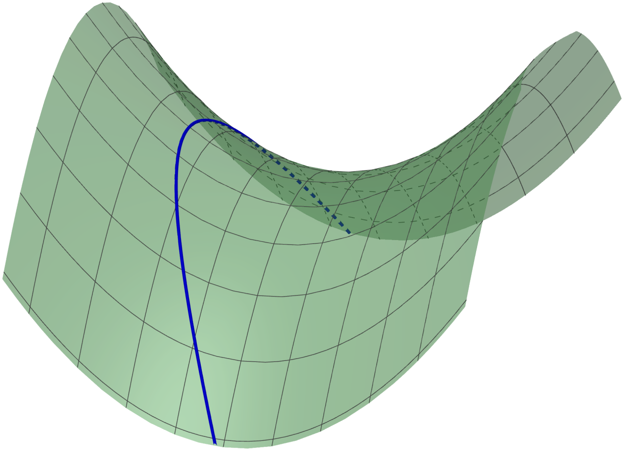
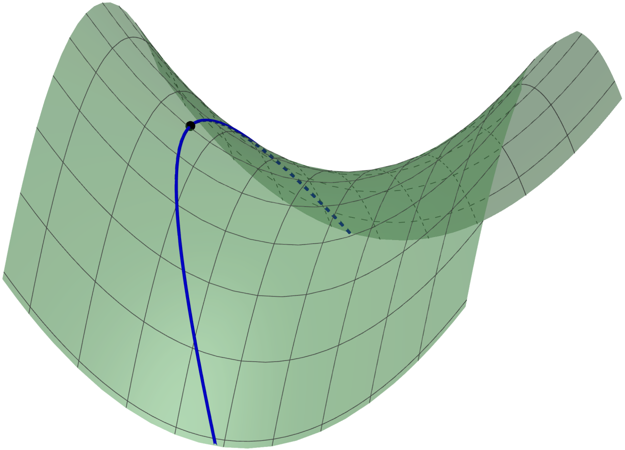
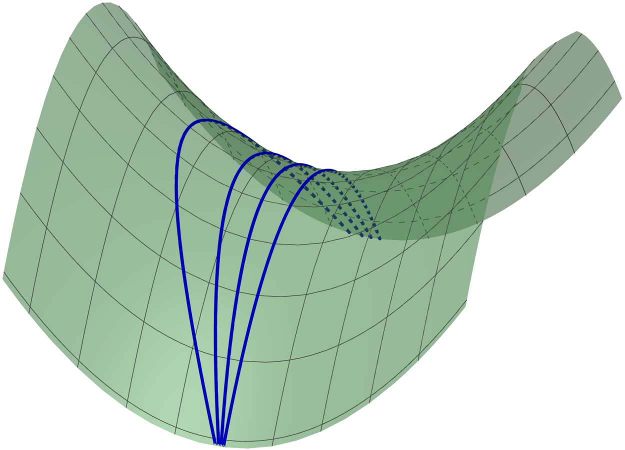
A sequence $\{\Sigma_t\}^n$ is **minimizing** if $\mathcal{F}(\{\Sigma_t\}^n) \to m_0(\Lambda)$.

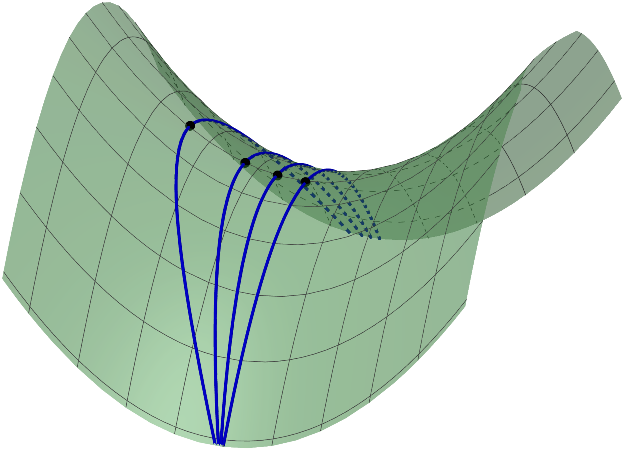
A sequence of slices $\{\Sigma^n_{t_n}\}$ is a **min-max sequence** if $\mathcal{H}^2(\Sigma^n_{t_n}) \to m_0(\Lambda)$.

To obtain minimal surfaces, we need to find a Λ such that $m_0(\Lambda) > 0$.









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Let $U_t = f^{-1}([0,t))$, $V_t = \psi(t, U_t)$. The volume $\mathrm{Vol}(V_t)$ is continuous, and moreover $\mathrm{Vol}(V_0) = 0$, $\mathrm{Vol}(V_1) = \mathrm{Vol}(M)$.

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In particular, there exists V_s whose volume is $\mathrm{Vol}(M)/2$. By the isoperimetric inequality and by definition of $\mathcal{F}(\{\Gamma_t\})$,

$$0 < c(M) \leq \mathcal{H}^2(\Gamma_s) \leq \mathcal{F}(\{\Gamma_t\}),$$

and so $m_0(\Lambda) \geq c(M) > 0$.

Main Theorem

Theorem [Simon-Smith]

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We will find an appropriate min-max sequence converging to a **minimal surface**.

Table of Contents

Regular surfaces as linear operators

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Any surface $\Sigma \subset U$ with finite area induces a (non negative) bounded linear operator on $C_c(G(U))$:

$$\varphi(x, \pi) \in C_c(G(U)) \longmapsto \int_\Sigma \varphi(x, T_x\Sigma) d\mathcal{H}^2$$

Varifolds: a weak notion of surfaces

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Moreover, there exists a unique measure $\|V\|$, called the **mass measure**, defined on U, and such that given $\varphi \in C_c(U)$,

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If Σ is a surface and V_Σ is its associated varifold, then $\|V_\Sigma\|(U)$ is the area of Σ in U.

The space of varifolds and its compactness properties

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is **metrizable** and **compact**. In particular, any sequence of varifolds with uniformly bounded mass has a **convergent subsequence**.

Useful varifold concepts

A varifold V is **rectifiable** if there is a set $S = \cup_{i \in \mathbb{N}} \Sigma_i$, where Σ_i are C^1 surfaces, and a Borel function $h : S \to [0, \infty)$ such that

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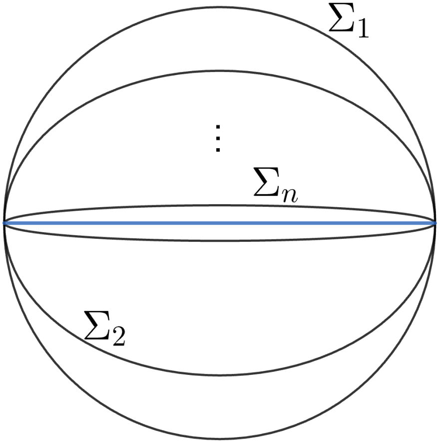
Given $x \in M$, $V \in \mathcal{V}(U)$ and $\rho > 0$, let T^x_ρ be the dilation of factor ρ, and define $V^x_\rho := \left(T^x_\rho\right)_\# V$. Any limit $V' = \lim_{n \to \infty} V^x_{\rho_n}$ with $\rho_n \to \infty$ is a **tangent varifold** to V at x. We denote the set of all these limits by $T(x, V)$.

Some examples

Let $\{\Sigma_n\}$ be a family of spheres that we *flatten*.



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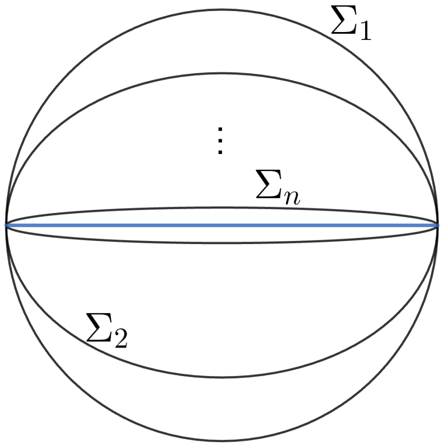
In the limit, this family converges to a disk (with multiplicity two).



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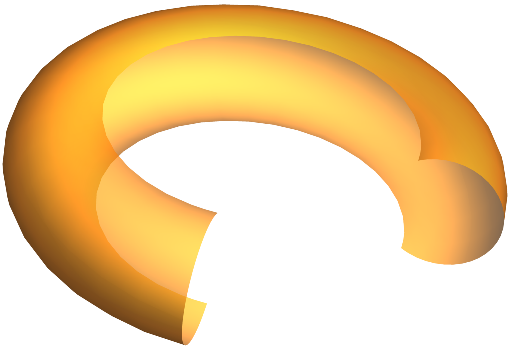

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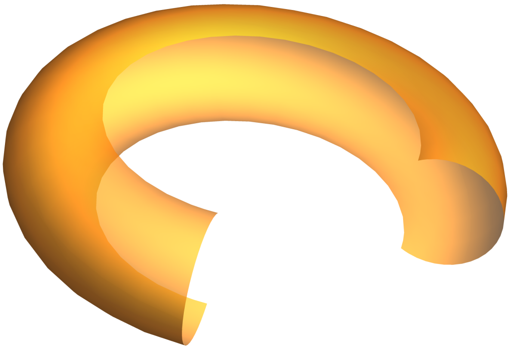

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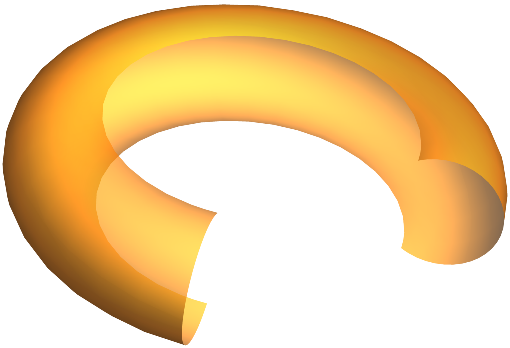

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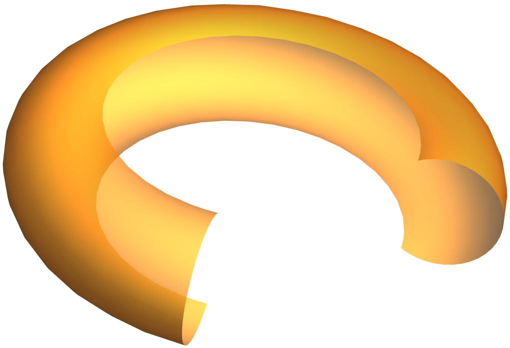
In the limit, $\{\Sigma_n\}$ converge to a torus with multiplicity 2. In particular, **genus can be increased**.



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As measures on $\mathbb{R}^2$, these segments converge to a horizontal segment with multiplicity $\sqrt{5}!$.

However, as **varifolds**, S_n converge to a **non-rectifiable** varifold.

First variation and stationary varifolds

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$$[\delta V](\chi) = \frac{d}{dt}(\|\psi_\chi(t,\cdot)_\# V\|)\Big|_{t=0} = \int_{G(U)} \mathrm{div}_\pi \chi \, dV(x,\pi).$$

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Stationary varifolds have several **key properties**:

- It is *not hard* for a stationary varifold to be rectifiable.
- They satisfy some minimal properties: monotonicity formula, a *maximum principle*...

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Theorem (Pull-tight process)

There exists a **minimizing sequence** $\{\Gamma_t\}^n$ such that **any** min-max sequence $\{\Gamma^n_{t_n}\}$ converges to a stationary varifold.

Idea of the Theorem: For each varifold we will define an isotopy $\psi_V(t, x)$ such that:

- If V is stationary, then $\psi_V(t, \cdot)$ is the identity map.
- Otherwise, $V' := (\psi_V(1, \cdot))_\# V$ has strictly less mass than V.
- The difference between $\|V'\|(M)$ and $\|V\|(M)$ depends uniformly on the distance between V and the set of stationary varifolds.

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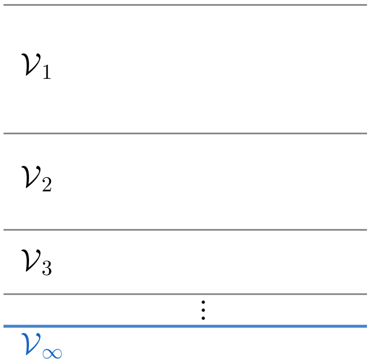
Let $\mathcal{V}_\infty \subset X$ be the (compact) set of **stationary varifolds** in X, and

$$\mathcal{V}_k := \{V \ : \ 2^{-k+1} \geq \mathfrak{d}(V, \mathcal{V}_\infty) \geq 2^{-k}\}$$

The sets $\mathcal{V}_k$ are also **compact**.



Let X be the set of varifolds with mass less or equal than $4m_0$, which is **compact** and **metrizable**.

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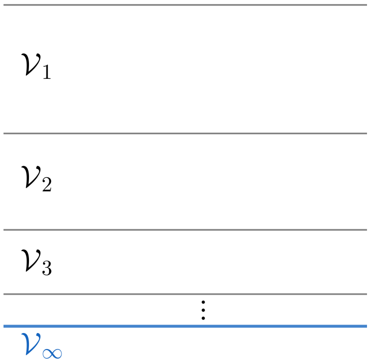
The sets $\mathcal{V}_k$ are also **compact**.

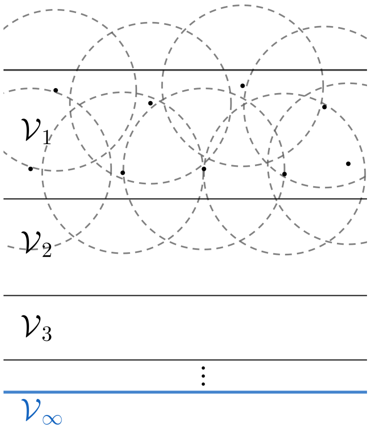
There exists $c = c(k) > 0$ such that, for each $V \in \mathcal{V}_k$ there exists χ_V with $\|\chi_V\| \leq 1$ and

$$[\delta V](\chi_V) \leq -c(k).$$



By **compactness**, we can take a finite set of varifolds $V_i^k \subset \mathcal{V}_k$, fields χ_i^k and balls $B(V_i^k, r_i^k)$ s.t.:

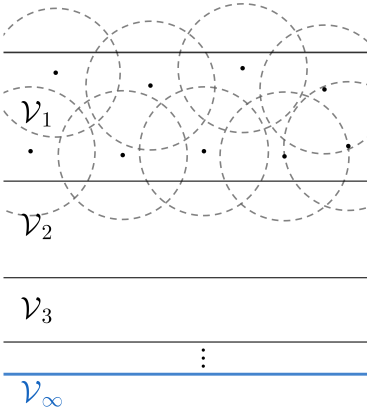
- If $V \in B(V_i^k, r_i^k)$, then $[\delta V](\chi_i^k) \leq -c(k)/2$.



Pull-tight process

By **compactness**, we can take a finite set of varifolds $V_i^k \subset \mathcal{V}_k$, fields χ_i^k and balls $B(V_i^k, r_i^k)$ s.t.:

- If $V \in B(V_i^k, r_i^k)$, then $[\delta V](\chi_i^k) \leq -c(k)/2$.
- $B(V_i^k, r_i^k/2)$ cover $\mathcal{V}_k$.



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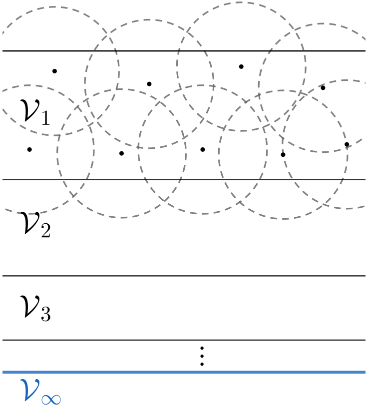
Let $\varphi_i^k \in C_c(B(V_i^k, r_i^k/2))$ be a partition of the unit, and define $H : X \to C^\infty(M, TM)$ as

$$H(V) := \sum_{i,k} \varphi_i^k(V)\chi_i^k.$$

Notice that H is **continuous**.

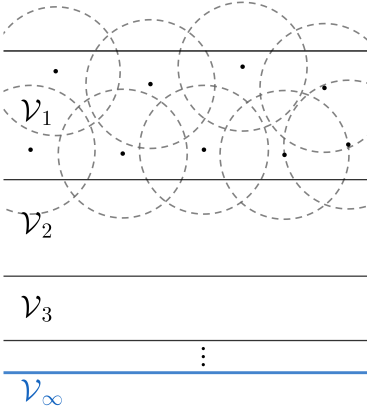


Pull-tight process

There exist constants $C = C(k)$ and a time $T = T(k)$ such that

$$\|V(T)\|(M) \leq \|V(0)\|(M) - C(k)$$

for every $V \in \mathcal{V}_k$, where $V(T)$ is the *evolution* of $V = V(0)$ under the field $H(V)$.



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Now, let $\{\Sigma_t\}^n$ be a minimizing sequence. Then, we can define a *tighter* minimizing sequence:

Let $\Gamma_t^n := \Sigma_t^n(T)$. Then, the sequence $\{\Gamma_t\}^n$ is minimizing and each min-max sequence converges to a **stationary varifold**.

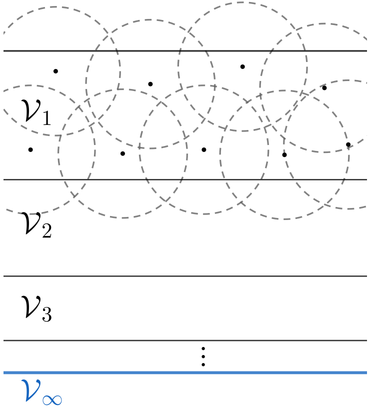

Table of Contents

Almost minimizing surfaces

Let $\varepsilon > 0$, $U \subset M$ and $\Sigma \subset U$. We will say that the surface Σ is ε-almost minimizing if there **does not** exist any isotopy ψ supported in U such that:

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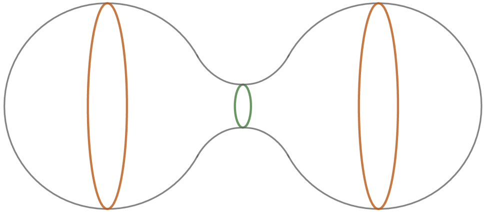
Let $\mathcal{AN}(x, r)$ be the set of annuli centered in $x \in M$ with *outer radius* less than r.

A sequence $\{\Sigma^n\}$ is almost minimizing in small annuli if there exists $r : M \to (0, \infty)$ such that $\{\Sigma^n\}$ is almost minimizing in every $An \in \mathcal{AN}(x, r(x))$.

An example

Consider the following dumbbell.



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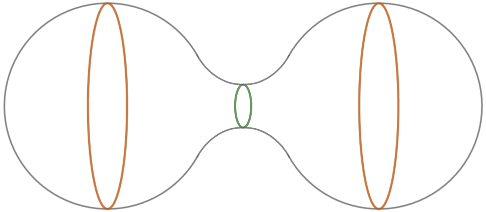
The three geodesics could be deformed to a point, so **they are not minimizing**.



An example

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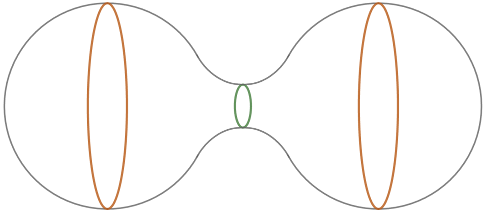
The three geodesics could be deformed to a point, so **they are not minimizing**.

However, the geodesic in the middle is almost minimizing: in order to decrease its length, one must increase it first.



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It is natural to ask why we consider minimization on annuli rather than balls. There are two reasons for it:

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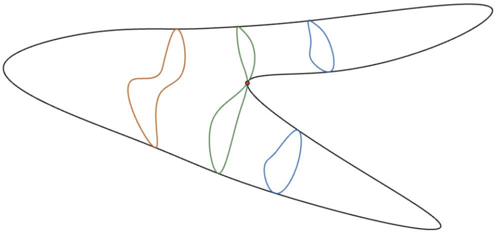

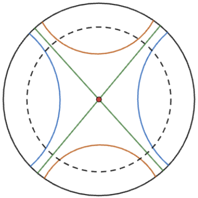

Replacements

Let V be a stationary varifold and $U \subset M$. We say that V' is a **replacement** of V in U if:

- V' is stationary and $\|V\| = \|V'\|$.
- $V = V'$ on $M \setminus U$.
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We say that V has the **good replacement property** if:

- V has a replacement V' in any $An \in \mathcal{AN}(x, r(x))$.
- V' has a second replacement V'' in any $An \in \mathcal{AN}(y, r(y))$.
- V'' has a third replacement V''' in any $An \in \mathcal{AN}(z, r(z))$.

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Theorem 3 (a.m. min-max sequence has GRP)

The varifold V of the previous Proposition has the good replacement property. In particular, V is an **embedded, minimal surface** with area $m_0(\Lambda)$.

Talk structure

Lemma

If V is stationary and admits replacements in small annuli, then it is **integer rectifiable**. Moreover, its tangent varifolds are **planes**.

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Given a **stationary** varifold V, assume that $\theta(x,V) > 0$ $\|V\|$-a.e. where $\theta(x,V) = \lim_{r\to 0} f_x^V(r)$. Then, V is **rectifiable**. [GMT, Theorem 8.1.2.]

Now we prove that V is **integer** rectifiable:

Let V be a stationary rectifiable varifold with $\theta(x, V) \geq c > 0$ for all $x \in M$. Then, any tangent varifold is a Euclidean stationary rectifiable cone. [GMT, Corollary 8.5.7]

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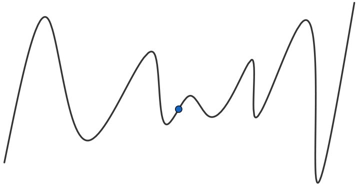
Schoen's curvature estimate

If $\{\Sigma^n\}$ is a sequence of stable minimal surfaces in $U \subsetneq M$, then a subsequence converges to a stable minimal surface Σ.

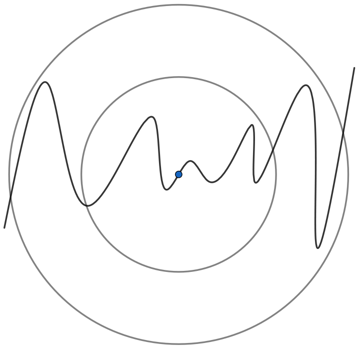
Replacements of V

If V is stationary and admits replacements in small annuli, then it is **integer rectifiable**. Moreover, its tangent varifolds are **planes**.



If V is stationary and admits replacements in small annuli, then it is **integer rectifiable**. Moreover, its tangent varifolds are **planes**.

We fix some $x \in M$, $\rho > 0$ and consider the annulus $An_1 := An(x, \rho, 2\rho)$.



Replacements of V

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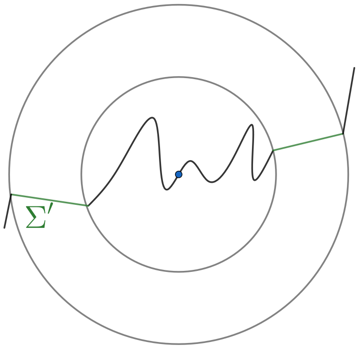
We fix some $x \in M$, $\rho > 0$ and consider the annulus $An_1 := An(x, \rho, 2\rho)$.

We take a good replacement V' of V in An_1, and define $\Sigma' := V'|_{An_1}$.



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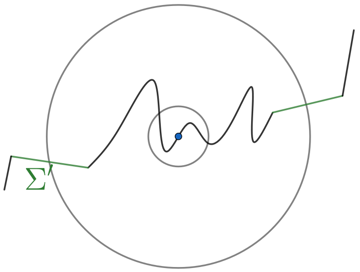
We take a good replacement V' of V in An_1, and define $\Sigma' := V'|_{An_1}$.

Given $s < \rho < t < 2\rho$, we consider a further replacement V'' of V' in $An_2(s) := An(x, s, t)$, and $\Sigma'' := V|_{An_2}$.



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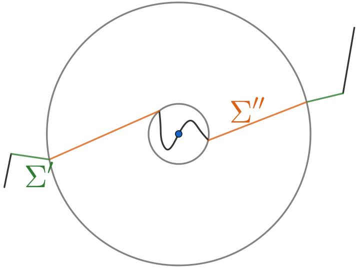

If V is stationary and admits replacements in small annuli, then it is **integer rectifiable**. Moreover, its tangent varifolds are **planes**.

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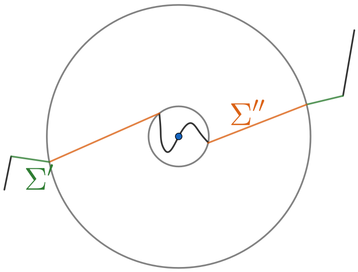
We have two surfaces Σ' and Σ''.



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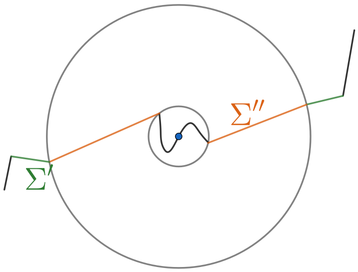
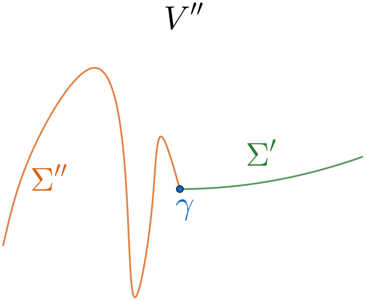
We have two surfaces Σ' and Σ''. **Do they coincide?**



Let $\gamma := \Sigma' \cap \partial B_t$, and take t so that Σ' meets ∂B_t transversally. One can check that V'' has a tangent plane along γ, which must coincide with that of Σ'.



Gluing two minimal surfaces

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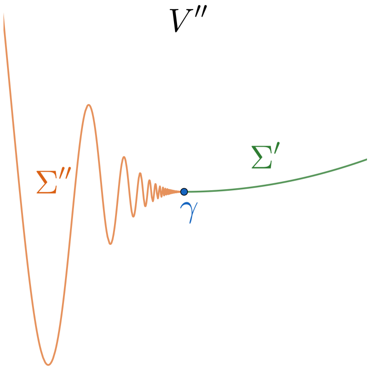
From this, we deduce that *boundary* of Σ'' meets Σ' *tangentially*.



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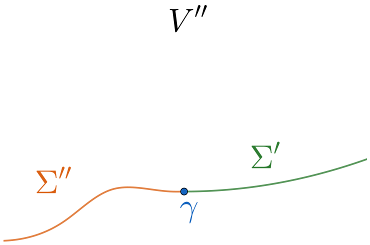

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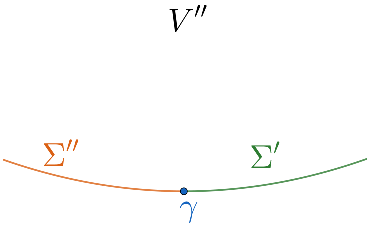
By PDE theory, since Σ', Σ'' and their Gauss maps coincide along γ, $\Sigma' \equiv \Sigma''$.



Extending Σ

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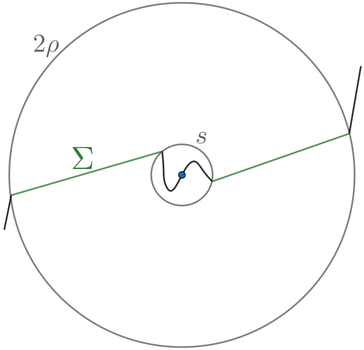

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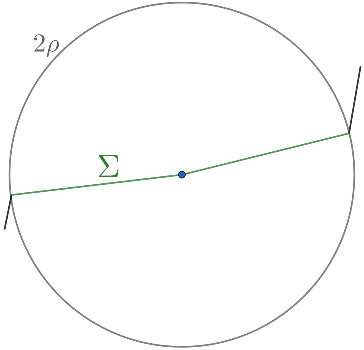
Let $\Sigma := \Sigma' \equiv \Sigma''$. By our previous discussion, Σ can be extended to any $An(x, s, 2\rho)$, and even to $B^*(x, 2\rho)$.



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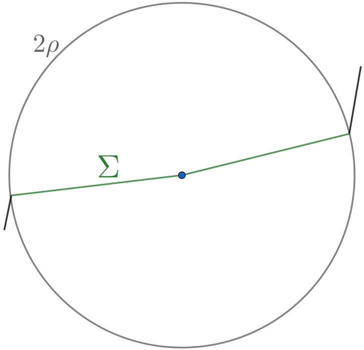

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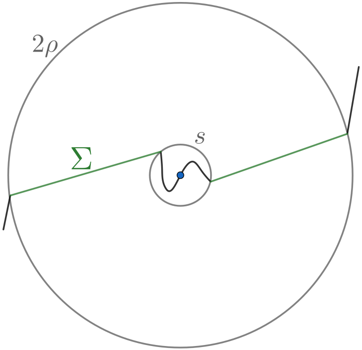

Extending Σ

Let $\Sigma := \Sigma' \equiv \Sigma''$. By our previous discussion, Σ can be extended to any $An(x, s, 2\rho)$, and even to $B^*(x, 2\rho)$. In fact,

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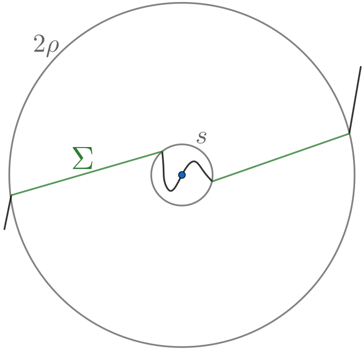

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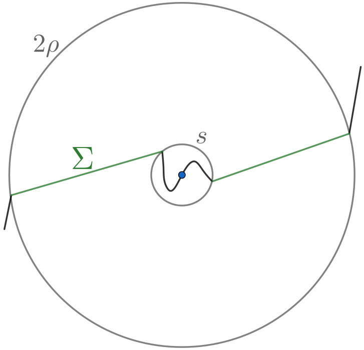
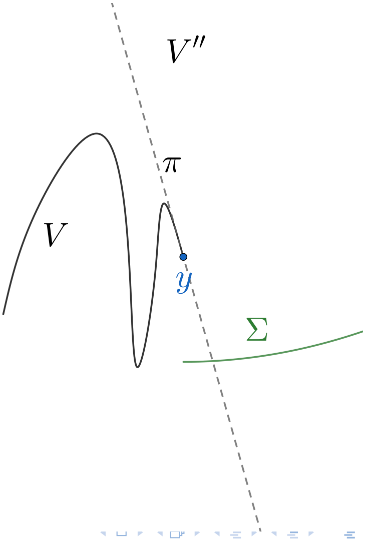
Let $y \in \text{Supp}(\|V\|) \cap B^*(x, \rho)$ s.t. V meets $\partial B(x, s)$ **transversally**, where $s := d(x, y)$. In particular, there exists a plane $\pi \in T_y M$ tangent to V.



$V = \Sigma$ on a punctured ball

Since $V = V''$ in $B(x, s)$, π is tangent to V''.



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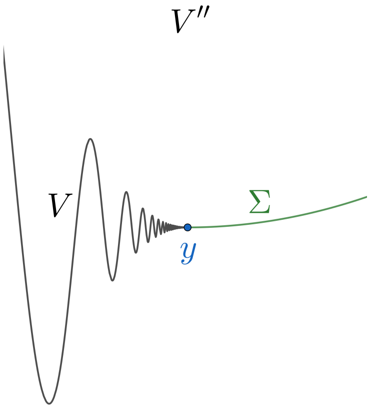

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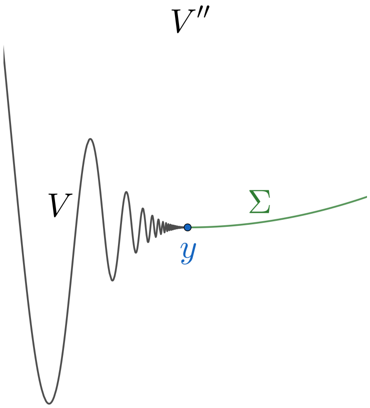

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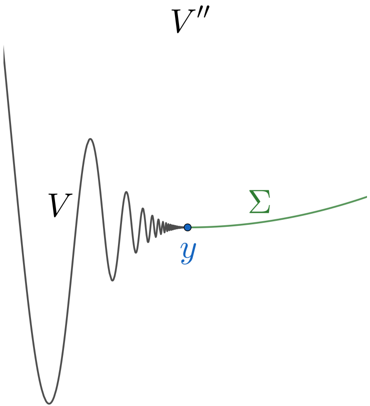

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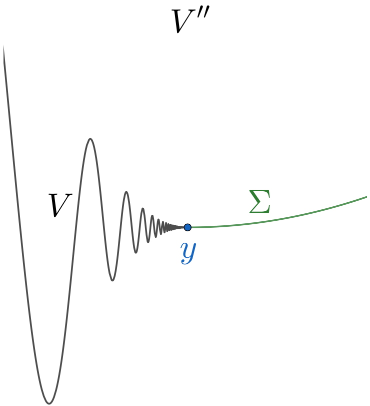

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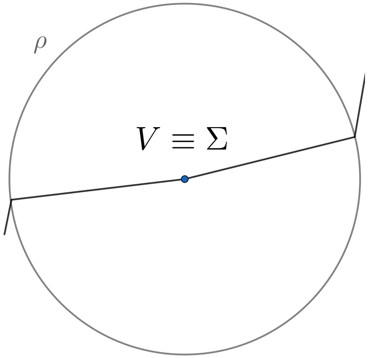

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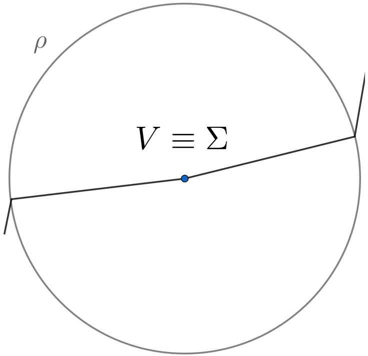
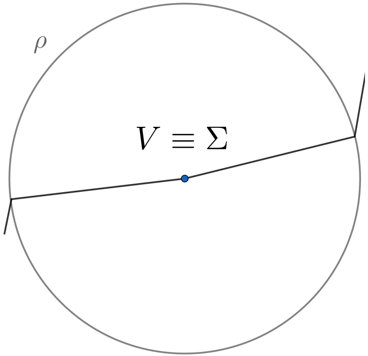
As a result, $V|_{B^*(x,\rho)} = \Sigma$. Now, can we extend Σ smoothly to x?



Smooth extension of Σ at x

We know that the tangent varifold to V at x is a **plane** π with multiplicity M.



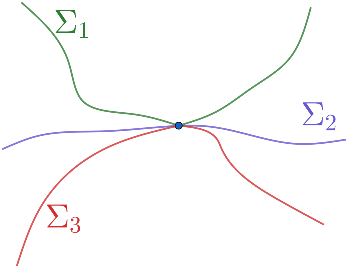
Smooth extension of Σ at x

We know that the tangent varifold to V at x is a **plane** π with multiplicity M.

Stability shows that near x, there are minimal Lipschitz graphs Σ_i and constants m_i, $1 \leq i \leq N$ with $\sum m_i = M$ and

$$\Sigma = \sum m_i \Sigma_i$$

near x. Moreover, Σ_i are ordered by height.



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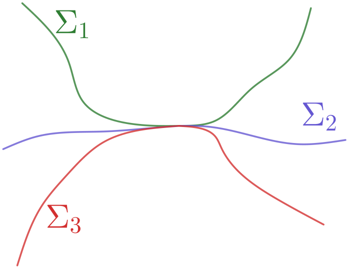
By Allard's regularity Theorem, each Σ_i extends smoothly to x.



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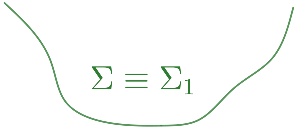
By Allard's regularity Theorem, each Σ_i extends smoothly to x. By the maximum principle, $N = 1$, and Σ is **embedded**.



Talk structure

Regularity results

Theorem 1 (GRP implies minimality)

If V has the good replacement property, then it is an **embedded minimal surface**.

Theorem 2 (Existence of a.m. min-max sequence)

There exists a min-max sequence $\{\Sigma^n\}$ which is **almost minimizing** in small annuli and converges to a **stationary varifold** V. Moreover, given any small annulus An, $\Sigma^n|_{An}$ is a **smooth surface** for sufficiently large n.

Theorem 3 (a.m. min-max sequence has GRP)

The varifold V of the previous Proposition has the good replacement property. In particular, V is an **embedded, minimal surface** with area $m_0(\Lambda)$.

Sketch of the proof

We need to prove that there exists a min-max sequence $\{\Sigma^n\}$ s.t.:

1. It converges to a stationary varifold,
2. For sufficiently small annuli An, $\Sigma^n|_{An}$ is a **smooth** surface,
3. It is almost minimizing in small annuli.

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We deduce that there is $r = r(x)$ such that every annulus An with outer radius less than $r(x)$ does not contain any P_i nor P_i^n for large n.

Finding an almost minimizing min-max sequence

Let $\mathcal{CO}$ be the set of pairs (U_1, U_2) such that

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Proof of Theorem 2: Consider the pairs $(B_r(x), M \setminus B_{4r}(x))$. Then, either

- there exists $r > 0$ s.t. a subsequence $\{\Sigma^{L(j)}\}$ is $1/L$-a.m. in $B_r(x)$ for all x,
- There is a subsequence $\{\Sigma^{L(j)}\}$, $r_j \to 0$ and $x_j \to x^*$ such that $\Sigma^{L(j)}$ is $1/L$-a.m. in $M \setminus B_{r_j}(x_j)$.

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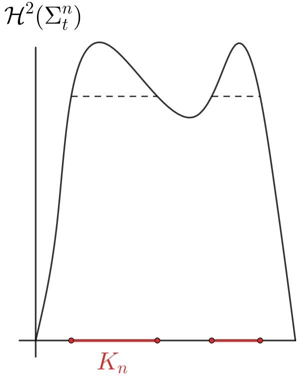
In any of the cases, we obtain a min max subsequence $\{\Sigma^{L(j)}\}$ which is a.m. in small annuli.

Proof of the Proposition

Let $L \in \mathbb{N}$, and

$$K_n := \{t \in [0,1] \; : \; \mathcal{H}^2(\Sigma_t^n) \geq m_0(\Lambda) - 1/L\}.$$



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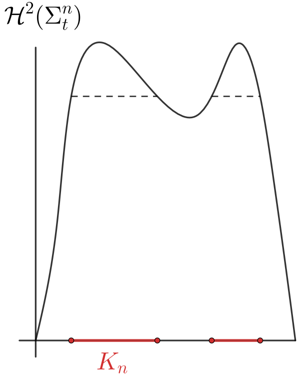

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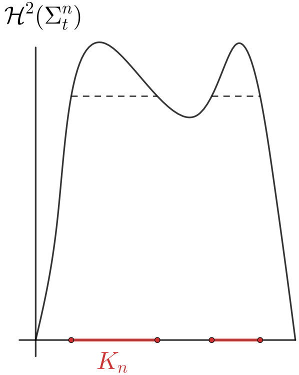

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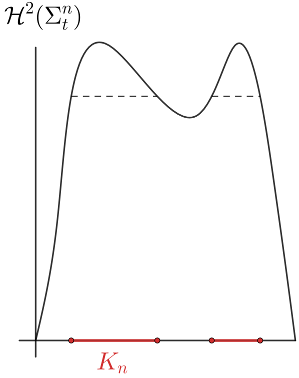
We can find at least two isotopies ψ_U and ψ_V which decrease the area of Σ_t^n with small increase in the process.



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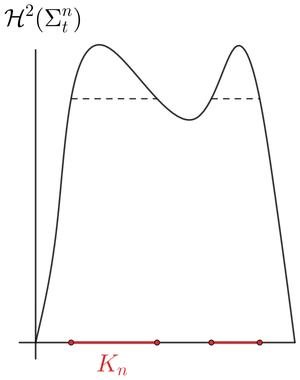

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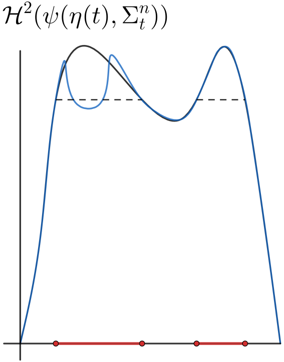
Idea: apply one of the isotopies ψ_U, ψ_V along I.



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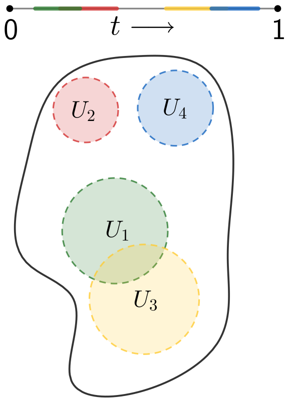
Now, we take a finite cover $\{I_j\}_{1\leq j\leq N}$ of K_n by some of these intervals, each associated with an isotopy $\psi_j(\eta_j,\cdot)$ supported in U_j.



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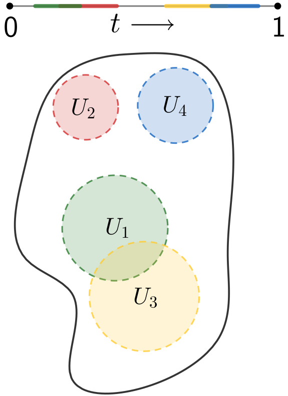
- $I_j \cap I_k = \varnothing$ unless $|j-k| = 1$.
- The sets U_j, U_{j+1} are **disjoint** if $I_j \cap I_{j+1}$ overlap.
- For all $t \in K_n$, one of the $\eta_j(t)$ is 1.



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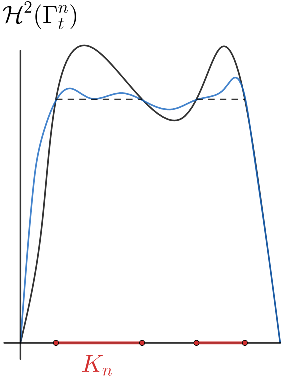

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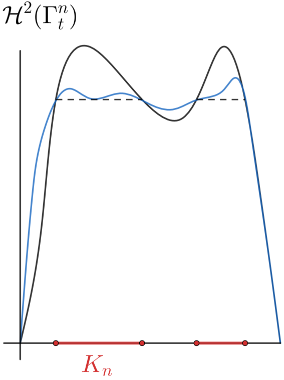

After applying the isotopies $\psi(\eta_j(t),\cdot)$ to $\{\Sigma_t^n\}$, we obtain a new family Γ_t^n satisfying:

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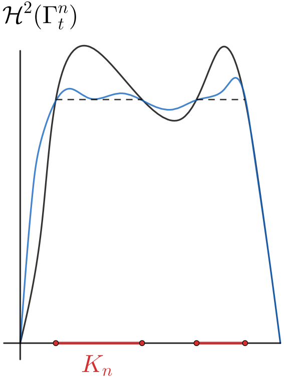
In particular, $\lim_n \mathcal{F}(\{\Gamma_t^n\}) \leq m_0(\Lambda) - 1/2L$!!



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Theorem 3 (a.m. min-max sequence has GRP)

The varifold V of the previous Proposition has the good replacement property. In particular, V is an **embedded, minimal surface** with area $m_0(\Lambda)$.

Let $\mathcal{I}$ be a set of smooth isotopies on M and Σ be a surface. We say that $\{\psi^k(1,\Sigma)\} \subset \mathcal{I}$ is **minimizing** for $(\Sigma, \mathcal{I})$ if

$$\lim_k \mathcal{H}^2(\psi^k(1,\Sigma)) = \inf_{\psi \in \mathcal{I}} \mathcal{H}^2(\psi(1,\Sigma)).$$

Preliminary definitions and results

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Given $U \subset M$ and an **embedded** surface Σ, we define:

- $\mathfrak{Is}(U)$: the set of all smooth isotopies supported in U.
- $\mathfrak{Is}_j(U) := \{\psi \in \mathfrak{Is}(U) \ : \ \mathcal{H}^2(\psi(t,\Sigma)) \leq \mathcal{H}^2(\Sigma) + \frac{1}{8j}\}$.

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Theorem (Meeks-Simon-Yau)

Let $\{\Sigma^k\} \subset \mathfrak{Is}(U)$ be **minimizing** and converging to a varifold V. Then, $V|_U$ is an **stable, embedded, minimal surface**.

Proof of Theorem 3

Let $\{\Sigma^j\}$ be the a.m. min-max sequence obtained in Theorem 2, which converges to a stationary varifold V.

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Let $An \in \mathcal{AN}(x, r(x))$ be a small annulus. For each j, let $\{\Sigma^{j,k}\}^k$ be **minimizing** for $(\Sigma^j, \mathfrak{Is}_j(An))$ and converging to a varifold V^j. Then, $V^j|_{An}$ is a **stable, embedded, minimal surface**.

Any limit V^* of a subsequence of $\{V^j\}$ is a replacement for V in An.

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- V^* is stationary: Otherwise, one can prove that Σ^j cannot be $1/j$-almost minimizing for j big enough.

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Fix j, and let $V' := V^j$ be the limit of $\Sigma^k := \Sigma^{j,k}$. We want to prove that $V'|_{An}$ is a stable, embedded, minimal surface. The proof is based on Lemma A:

Let $x \in An$ and k large enough. There exists $\varepsilon > 0$ s.t. any isotopy $\varphi \in \mathfrak{Is}(B_\varepsilon(x))$ can be achieved via an isotopy $\Phi \in \mathfrak{Is}_j(B_{2\varepsilon}(x))$.

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We can now define a diagonal sequence $\{\Sigma^{k,l(k)}\}^k$ converging to W. Applying Lemma A, we obtain a second replacement.

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We can now define a diagonal sequence $\{\Sigma^{k,l(k)}\}^k$ converging to W. Applying Lemma A, we obtain a second replacement. Iterating this process, we deduce the good replacement property.

Proof of Lemma A

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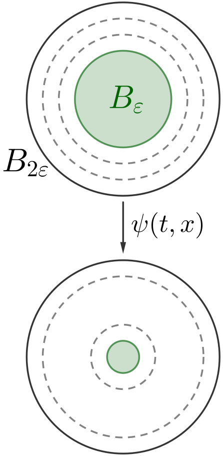
Proof: Let $\psi(t, x)$ be an isotopy which *squeezes* the ball $B_\varepsilon(x)$ in $B_{2\varepsilon}(x)$ with scale factor $(1 - t)$.



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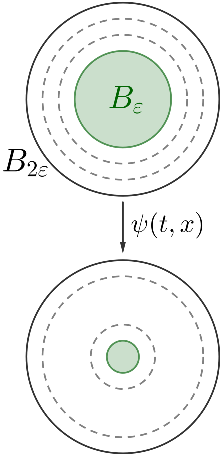
Proof: Let $\psi(t, x)$ be an isotopy which *squeezes* the ball $B_\varepsilon(x)$ in $B_{2\varepsilon}(x)$ with scale factor $(1 - t)$.

For large enough k, we can find a certain $\psi(t, x)$ satisfying

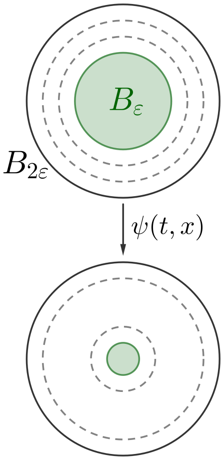
$$\mathcal{H}^2(\psi(t, \Sigma^k)) \leq \mathcal{H}^2(\Sigma^k) + C\varepsilon^2.$$



Proof of Lemma A

Now, given $\varphi \in \mathfrak{Is}(B_\varepsilon(x))$, let Φ be the isotopy which applies this process:

- First, it *squeezes* B_ε via $\psi(t,x)$ up to a certain factor $(1 - t_0)$.
- Then, it applies the isotopy φ on the *squeezed ball* $B_{(1-t_0)\varepsilon}(x)$.
- Finally, it *enlarges* $B_\varepsilon(x)$ by applying $\psi(t,x)$ reversely.



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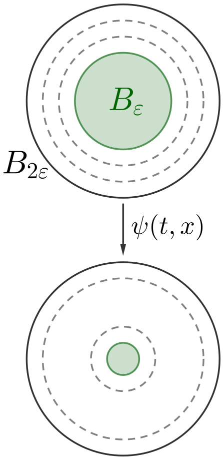
$\varphi(1,x) \equiv \Phi(1,x)$. Moreover, if t_0 is sufficiently close to 1, $\Phi(t,x) \in \mathfrak{Is}_j(B_{2\varepsilon}(x))$.



Regularity results

Theorem 1 (GRP implies minimality)

If V has the good replacement property, then it is an **embedded minimal surface**.

Theorem 2 (Existence of a.m. min-max sequence)

There exists a min-max sequence $\{\Sigma^n\}$ which is **almost minimizing** in small annuli and converges to a **stationary varifold** V. Moreover, given any small annulus An, $\Sigma^n|_{An}$ is a **smooth surface** for sufficiently large n.

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Idea: to construct sweepouts by using Heegaard surfaces.

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Idea: to construct sweepouts by using Heegaard surfaces.

A **Heegaard sweepout** is a family $\{\Sigma_t\}_{t\in[0,1]}$ of closed subsets of M such that:

- $t \mapsto \Sigma_t$ is continuous in the Hausdorff topology.
- For $t \in (0,1)$, Σ_t is a smooth Heegaard surface.
- Σ_t varies smoothly for $t \in (0,1)$.
- Σ_0 and Σ_1 are 1-d graphs which correspond to the spines of the handlebodies determined by the Heegaard surfaces.

Genus bounds in Simon-Smith min-max theory

Theorem (Ketover)

Let M be a oriented 3-manifold admitting a **genus g** Heegaard surface. Then, there exists some min-max sequence $\{\Sigma^j\}$ converging as varifolds to $\Gamma := \sum_j n_j \Gamma_j$, where Γ_j are pairwise disjoint, embedded, minimal surfaces. Moreover, it holds

$$\sum_{i \in O} n_i g(\Gamma_i) + \frac{1}{2} \sum_{i \in N} n_i \left(g(\Gamma_i) - 1 \right) \leq g,$$

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As a direct corollary,

Theorem (Simon-Smith)

Every **3-sphere** admits an embedded, minimal **2-sphere**.

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Moreover, we define the singular locus $\mathcal{S}$ of G as

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Moreover, we define the singular locus $\mathcal{S}$ of G as

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$\mathcal{S}$ can be expressed as a graph $\mathcal{S} := \mathcal{S}_0 \sqcup \mathcal{S}_1$, where $\mathcal{S}_1$ are **geodesics** whose endpoints belong to $\mathcal{S}_0$.

Equivariant min-max theory

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Let M be a closed oriented 3-manifold and G be a finite group of orientation-preserving isometries on M. Given a G-sweepout by surfaces of genus g, there exists an **embedded, minimal** G-equivariant surface $\Gamma = \sum_j n_j \Gamma_j$.

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Moreover, the following properties hold:

- The same genus bounds as in the previous Theorem.
- If $x \in \mathcal{S}_1 \cap \Gamma$ and $G_x \neq \mathbb{Z}_2$, then Γ intersects $\mathcal{S}_1$ **orthogonally** at x.
- If, however, $G_x = \mathbb{Z}_2$, the intersection may also be tangent. In that case, Γ contains the geodesic of constant isotropy passing through x.
- Γ can only intersect $\mathcal{S}_0$ at a point with isotropy group $\mathbb{D}_n$. In that case, Γ contains $2n$ arcs of isotropy $\mathbb{Z}_2$.

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What can we say about the boundary components of Γ?

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Let us assume that ∂M is **mean-convex**. Then,

Theorem (Franz, Schulz)

Given a G-sweepout by genus g (orientable) surfaces with b boundary components, there exists a **free boundary, embedded, minimal** surface $\Gamma = \sum_j n_j \Gamma_j$.

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Moreover, we have the genus bound:

$$\sum_{i \in O} g(\Gamma_i) + \frac{1}{2} \sum_{i \in N} (g(\Gamma_i) - 1) \leq g$$

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Regarding the boundary complexity $\mathfrak{b} := b - 1$,

$$g(\Gamma) + \mathfrak{b}(\Gamma) \leq g + \mathfrak{b}.$$

A general existence theorem

Given an open subset $\Omega \subset M$, let us consider the functional

$$\mathcal{A}(\Omega) := \mathcal{H}^2(\partial\Omega_1) + \cos(\theta)\mathcal{H}^2(\partial\Omega_2) - c\mathrm{Vol}(\Omega),$$

where $\partial\Omega = \partial\Omega_1 \sqcup \partial\Omega_2$, $\partial\Omega_2 := \partial\Omega \cap \partial M$.

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In general, critical points of $\mathcal{A}$ are **capillary, CMC** surfaces with $H \equiv c$ and contact angle θ.

Theorem (Li, Zhou, Zhu)

Given any closed oriented manifold M with smooth boundary, $c \geq 0$ and $\theta \in (0, \pi/2]$, there exists a **capillary, CMC**, almost embedded surface with contact angle θ and $H \equiv c$.

Note: No genus or boundary component bounds are known for this case.

Thank you for your attention!

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